7. Integracija racionalnih f-ja

Funkciju oblika $\frac{P(x)}{Q(x)}$, gdje su $P(x)$ i $Q(x)$ polinomi, nazivamo RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ manji od stepena polinoma $Q(x)$, tada kažemo da je f-ja $\frac{P(x)}{Q(x)}$ PRAVA RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ veći ili jednak od stepena polinoma $Q(x)$, tada je funkcija $\frac{P(x)}{Q(x)}$ NEPRAVA RACIONALNA F-JA.

Ako je racionalna f-ja $\frac{P(x)}{Q(x)}$ neprava, tada dijeljenjem polinoma $P(x)$ sa polinomom $Q(x)$ dobijemo količnik polinom $K(x)$, i ostatak dijeljenja, polinom $R(x)$, tako da je

$$P(x) = Q(x) \cdot K(x) + R(x) \quad \text{tj.} \quad \frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)}$$

Kako je stepen polinoma ostatka $R(x)$ manji od stepena polinoma djelioca $Q(x)$, slijedi da je $\frac{R(x)}{Q(x)}$ prava racionalna f-ja.

Pravu racionalnu f-ju možemo integrirati <u>metodom neodređenih koeficijenata</u>. To činimo na sljedeći način: Najprije polinom u nazivniku racionalne f-je rastavimo na proste faktore oblika

$$(x-a)^n \quad i \quad (x^2+px+q)^n \quad \text{gdje su}$$

$$n \in \mathbb{N}, \quad a, p, q \in \mathbb{R}, \quad p^2-4q<0.$$

Svakom faktoru oblika $(x-a)^n$ pridružimo f-ju oblika

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

($A_1, A_2, \dots, A_n$ su konstante koje trebamo odrediti),

a svakom faktoru oblika $(x^2+px+q)^n$ pridružimo f-ju oblika

$$\frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \dots + \frac{M_nx+N_n}{(x^2+px+q)^n}$$

gdje su $M_1, M_2, \dots, M_n$ i $N_1, N_2, \dots, N_n$ konstante koje treba odrediti.

Navedene racionalne f-je nazivamo parcijalni razlomci. Prema tome, svaku pravu racionalnu f-ju najprije rastavimo na zbir parcijalnih razlomaka, a zatim svaki od njih posebno integriramo. Kod neprave racionalne f-je najprije vršimo dijeljenje polinoma u brojniku sa polinomom u nazivniku,

Odrediti integrale

a) $\int \frac{3x^2+8}{x^3+4x^2+4x}dx$; b) $\int \frac{2x^5+6x^3+1}{x^4+3x^2}dx$;

c) $\int \frac{x^3+4x^2+4x}{x^4+x}dx$; d) $\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$.

Rj.

a) $\int \frac{3x^2+8}{x^3+4x^2+4x}dx$

Nazivnik rastavljamo na faktore.

$$x^3+4x^2+4x = x(x^2+4x+4) = x(x+2)^2$$

Poslije ovoga podintegralnu f-ju rastavljamo na sumu

$$\frac{3x^2+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Da bi odredili koeficijente A, B, C čitavu jednakost množimo sa $x(x+2)^2$ nakon čega dobijamo

$$3x^2+8 = A\underbrace{(x+2)^2}_{x^2+2x+2} + Bx(x+2) + Cx$$

$$= (A+B)x^2 + (4A+2B+C)x + 4A$$

Sad izjednačavamo koeficijente koji stoje uz x^2, x i x^0.

x^2: $A+B=3$ $\Rightarrow B=1$

x: $4A+2B+C=0$ $\Rightarrow C=-10$

x^0: $4A=8$ $\Rightarrow A=2$

Prema tome dobili smo $\frac{3x^2+8}{x(x+2)^2} = \frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}$

Sad možemo riješiti dati integral

$$\int \frac{3x^2+8}{x^3+4x^2+4x}dx = \int\left(\frac{2}{x}+\frac{1}{x+2}-\frac{10}{(x+2)^2}\right)dx =$$

$$= 2\int\frac{dx}{x} + \int\frac{dx}{x+2} - 10\int (x+2)^{-2}\, d(x+2) =$$

$$= 2\ln|x| + \ln|x+2| + \frac{10}{x+2} + C$$

b) $\int \frac{2x^5+6x^3+1}{x^4+3x^2}dx$

Podjelimo $2x^5+6x^3+1$ sa x^4+3x^2.

$$\begin{array}{l} (2x^5+6x^3+1):(x^4+3x^2) = 2x \\ \underline{-\ 2x^5+6x^3\qquad} \\ \qquad\qquad 1 \end{array}$$

Prema tome

$$\frac{2x^5+6x^3+1}{x^4+3x^2} = 2x + \frac{1}{x^4+3x^2}$$

Rastavimo nazivnik na faktore

$$x^4+3x^2 = x^2(x^2+3)$$

Napišimo ostatak $\frac{1}{x^4+3x^2}$ u obliku sume

$$\frac{1}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$$

Da bi smo odredili koeficijente A, B, C i D pomnožimo dobijenu jednakost sa $x^2(x^2+3)$:

$$1 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 =$$

$$= (A+C)x^3 + (B+D)x^2 + 3Ax + 3B$$

Izjednačimo brojeve koji stoje uz x^3, x^2, x i x^0:

$$\begin{array}{lll} x^3: & A+C=0 & \Rightarrow C=0 \\ x^2: & B+D=0 & \Rightarrow D=-\frac{1}{3} \\ x: & 3A=0 & \Rightarrow A=0 \\ x^0: & 3B=1 & \Rightarrow B=\frac{1}{3} \end{array}$$

$$\frac{1}{x^2(x^2+3)}=\frac{1}{3x^2}-\frac{1}{3(x^2+3)}$$

Sad nije teško izračunati dati integral

$$\int\frac{2x^5+6x^3+1}{x^4+3x^2}dx=\int\left(2x+\frac{1}{x^4+3x^2}\right)dx=$$

$$=\int\left(2x+\frac{\frac{1}{3}}{x^2}-\frac{\frac{1}{3}}{x^2+3}\right)dx=2\int x\,dx+\frac{1}{3}\int x^{-2}dx$$

$$-\frac{1}{3}\int\frac{dx}{x^2+3}=x^2-\frac{1}{3x}-\frac{1}{3\sqrt{3}}\operatorname{arc\,tg}\frac{x}{\sqrt{3}}+C$$

c) $$\int\frac{x^3+4x^2+4x}{x^4+x}dx$$

Rastavimo nazivnik x^4+x na faktore

$$x^4+x=x(x^3+1)=x(x+1)(x^2-x+1)$$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}\quad /\cdot x(x+1)(x^2-x+1)$$

$$x^3+4x^2-2x+1=A(x^3+1)+Bx(x^2-x+1)+(Cx+D)(x^2+x)$$

$$=(A+B+C)x^3+(C+D-B)x^2+(B+D)x+A$$

$$\begin{aligned} A+B+C&=1\\ -B+C+D&=4\\ B+D&=-2\\ A&=1 \end{aligned}$$

$A=1;\ B=-2;\ C=2;\ D=0$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)}=\frac{1}{x}-\frac{2}{x+1}+\frac{2x}{x^2-x+1}$$

Sad nije teško odrediti dati integral

$$I=\int\frac{x^3+4x^2-2x+1}{x^4+x}\,dx=\int\frac{dx}{x}-2\int\frac{dx}{x+1}+2\int\frac{x\,dx}{x^2-x+1}$$

$$=\ln|x|-2\ln|x+1|+2I_1$$

Da bi odredili I_1 imamo

$$x^2-x+1=x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$I_1=\int\frac{x\,dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}=\left|\begin{array}{l} x-\frac{1}{2}=t\\ dx=dt\end{array}\right|=\int\frac{\left(t+\frac{1}{2}\right)dt}{t^2+\frac{3}{4}}=\frac{1}{2}\int\frac{d\left(t^2+\frac{3}{4}\right)}{t^2+\frac{3}{4}}$$

$$+\frac{1}{2}\int\frac{dt}{t^2+\frac{3}{4}}=\frac{1}{2}\ln\left(t^2+\frac{3}{4}\right)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t}{\sqrt{3}}=$$

$$=\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}\,.$$

Prema tome imamo

$$I=\ln\frac{|x|\left(x^2-x+1\right)}{(x+1)^2}+\frac{2}{\sqrt{3}}\cdot\frac{2x-1}{\sqrt{3}}+C.$$

d) $\displaystyle\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$

$$x^4+10x^2+25=(x^2+5)^2$$

$$\frac{x^3-3}{(x^2-5)^2} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2} \qquad /\cdot(x^2-5)^2$$

$$x^3-3=(Ax+B)(x^2+5)+Cx+D$$

$$=Ax^3+Bx^2+(5A+C)x+(5B+D)$$

$$\begin{aligned} A&=1\\ B&=0\\ 5A+C&=0\\ 5B+D&=-3 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} A&=1\\ B&=0\\ C&=-5\\ D&=-3 \end{aligned}$$

$$\frac{x^3-3}{(x^2+5)^2} = \frac{x}{x^2+5} - \frac{5x+3}{(x^2+5)^2}$$

Sad imamo

$$\int \frac{x^3-3}{x^4+10x^2+25}\,dx = \underbrace{\int \frac{x\,dx}{x^2+5}}_{I_1} - 5\underbrace{\int \frac{x\,dx}{(x^2+5)^2}}_{I_2} - 3\underbrace{\int \frac{dx}{(x^2+5)^2}}_{I_3}$$

$$I_1=\int\frac{x\,dx}{x^2+5} = \frac{1}{2}\int\frac{d(x^2+5)}{x^2+5} = \frac{1}{2}\ln|x^2+5|$$

$$I_2=\int\frac{x\,dx}{(x^2+5)^2} = \frac{1}{2}\int(x^2+5)^{-2}d(x^2+5) = \frac{1}{2}\cdot\frac{(x^2+5)^{-1}}{-1} =$$

$$= -\frac{1}{2(x^2+5)}$$

$$I_3 = \int \frac{dx}{(x^2+5)^2} = \left| \begin{array}{l} x = \sqrt{5}\, tg\, z = \sqrt{5}\, \dfrac{\sin z}{\cos z} \\ dx = \sqrt{5}\, \dfrac{\cos^2 z + \sin^2 z}{\cos^2 z}\, dz = \dfrac{\sqrt{5}}{\cos^2 z}\, dz \\ x^2 = 5\, \dfrac{1-\cos^2 z}{\cos^2 z} \end{array} \right| =$$

$$= \int \frac{\dfrac{\sqrt{5}\, dz}{\cos^2 z}}{\left(\dfrac{5-5\cos^2 z}{\cos^2 z} + 5 \right)^2} = \sqrt{5} \int \frac{\dfrac{dz}{\cos^2 z}}{\dfrac{25}{\cos^4 z}} = \int \cos^2 z\, dz =$$

$$= \frac{\sqrt{5}}{25} \cdot \frac{1}{2} \int (1+\cos 2z)\, dz = \frac{\sqrt{5}}{50} \left(z + \frac{1}{2} \sin 2z \right)$$

$$= \frac{1}{10\sqrt{5}} \left(arc\, tg \frac{x}{\sqrt{5}} + \frac{x\sqrt{5}}{x^2+5} \right)$$

Prema tome

$$I = \frac{1}{2} \ln(x^2+5) + \frac{5}{2(x^2+5)} - \frac{3}{10\sqrt{5}} \left(arc\, tg \frac{x}{\sqrt{5}} + \frac{x\sqrt{5}}{x^2+5} \right) + C$$

$$= \frac{1}{2} \ln(x^2+5) + \frac{25-3x}{10(x^2+5)} - \frac{3}{10\sqrt{5}}\, arc\, tg \frac{x}{\sqrt{5}} + C$$

traženo rješenje

Zadaci za vežbu

Odrediti integrale

1. $\displaystyle\int \frac{dx}{x^3-x^2}$

2. $\displaystyle\int \frac{dx}{x^3+x}$

3. $\displaystyle\int \frac{x\,dx}{x^3-1}$

4. $\displaystyle\int \frac{(x^2+1)\,dx}{x^3-3x^2+3x-1}$

5. $\displaystyle\int \frac{(7x-15)\,dx}{x^3-2x^2+5x}$

6. $\displaystyle\int \frac{2t^5-2t+1}{1-t^4}\,dt$

7. $\displaystyle\int \frac{z^2\,dz}{z^4+5z^2+4}$

8. $\displaystyle\int \frac{x^4\,dx}{x^4-2x^2+1}$

9. * $\displaystyle\int \frac{(x+1)\,dx}{x^4+4x^2+4}$

10. * $\displaystyle\int \frac{1-x^4}{1+x^4}\,dx$

Rešenja:

1. $\frac{1}{x}+\ln\left|1-\frac{1}{x}\right|$
2. $\ln\frac{|x|}{\sqrt{x^2+1}}$
3. $\frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$
4. $\ln|x-1|-\frac{2x-1}{(x-1)^2}$
5. $3\ln\frac{\sqrt{x^2-2x+5}}{|x|}+2\operatorname{arctg}\frac{x-1}{2}$
6. $\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-t^2+\frac{1}{2}\operatorname{arctg}t$
7. $\frac{2}{3}\operatorname{arctg}\frac{z}{2}-\frac{1}{3}\operatorname{arctg}z$
8. $\frac{2x^3-3x}{2(x^2-1)}+\frac{3}{4}\ln\left|\frac{x-1}{x+1}\right|$
9. $\frac{x-2}{4(x^2+2)}+\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$
10. $-x+\frac{1}{2\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{1-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija racionalnih funkcija)

$g_m(x)$ - polinom stepena m

npr. $p(x) = 5x^7 - 3x^2 + x + 8$, polinom 7-og stepena

Racionalna f-ja je količnik dva polinoma.

$s(x) = \dfrac{p_n(x)}{g_m(x)}$. Za $n<m$ $s(x)$ je prava racionalna f-ja

Racionalnu f-ju razlažemo na proste razlomke.
Prosti razlomci su oblika:

$$\frac{A}{(ax+b)^n}, \quad \frac{Bx+C}{(ax^2+bx+c)^n}, \quad n \in \mathbb{N}$$

Izračunajte integrale:

1. $$I = \int \frac{x}{(x-1)(x+1)^2}\,dx$$

Rj. $$\frac{x}{(x-1)(x+1)^2} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{(x+1)^2} \quad / \cdot (x-1)(x+1)^2$$

$$x = a(x+1)^2 + b(x-1)(x+1) + c(x-1)$$

$$x = a(x^2+2x+1) + b(x^2-1) + c(x-1)$$

$$x = (a+b)x^2 + (2a+c)x + (a-b-c)$$

$$\Rightarrow \begin{array}{lr} a+b = 0 & (1) \\ 2a + c = 1 & (2) \\ a-b-c = 0 & (3) \end{array}$$

$$\begin{array}{ll} (1) & a+b=0 \\ (2)+(3): & 3a-b=1 \\ \hline & 4a=1 \end{array}$$

$a = \frac{1}{4}$, $b = -\frac{1}{4}$, $c = \frac{1}{2}$

$$\frac{x}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{2}}{(x+1)^2}$$

$$I = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{(x+1)^2} =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\,\frac{(x+1)^{-1}}{-1} + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2(x+1)} + C$$

(2.) $I=\int \frac{8x-31}{x^2-9x+14}\,dx$, $x^2-9x+14=0$ $\quad D=81-56 \quad D=25$ $\quad x_{1,2}=\frac{9\pm 5}{2}$ $\quad x_1=2,\ x_2=7$

$x^2-9x+14=(x-2)(x-7)$

Rj.
$$\frac{8x-31}{x^2-9x+14}=\frac{8x-31}{(x-2)(x-7)}=\frac{a}{x-2}+\frac{b}{x-7} \quad /(x-2)(x-7)$$

$$8x-31=a(x-7)+b(x-2)$$
$$8x-31=(a+b)x+(-7a-2b)$$

$$\begin{aligned} a+b&=8 \quad /\cdot 2\\ -7a-2b&=-31\\ \hline 2a+2b&=16\\ -7a-2b&=-31\\ \hline \end{aligned}$$

$-5a=-15$

$a=3$

$b=5$

$$\frac{8x-31}{x^2-9x+14}=\frac{3}{x-2}+\frac{5}{x-7}$$

$$I=3\int\frac{dx}{x-2}+5\int\frac{dx}{x-7}=3\ln|x-2|+5\ln|x-7|+C$$

(3.) $I=\int\frac{dx}{x^3-2x^2+x}$, $x^3-2x^2+x=x(x^2-2x+1)=x(x-1)^2$

Rj.
$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{a}{x}+\frac{b}{x-1}+\frac{c}{(x-1)^2} \quad /x(x-1)^2$$

$$1=a(x-1)^2+bx(x-1)+cx$$
$$1=a(x^2-2x+1)+b(x^2-x)+cx$$
$$1=(a+b)x^2+(-2a-b+c)x+a$$

$$\begin{aligned} a+b&=0\\ -2a-b+c&=0\\ a&=1\\ \hline \end{aligned}$$

$b=-1 \Rightarrow c=1$

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x}+\frac{-1}{x-1}+\frac{1}{(x-1)^2}$$

$$I=\int\frac{dx}{x}-\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C$$

(4.) $I=\int\frac{x^3+x+1}{x^4-1}\,dx$, $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

Rj.
$$\frac{x^3+x+1}{x^4-1}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{cx+d}{x^2+1} \quad /(x-1)(x+1)(x^2+1)$$

$$x^3+x+1 = a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x-1)(x+1)$$

$$x^3+x+1 = a(x^3+x+x^2+1) + b(x^3+x-x^2-1) + (cx+d)(x^2-1)$$

$$x^3+x+1 = a(x^3+x^2+x+1) + b(x^3-x^2+x-1) + c(x^3-x) + d(x^2-1)$$

$$x^3+x+1 = (a+b+c)x^3 + (a-b+d)x^2 + (a+b-c)x + (a-b-d)$$

$$\begin{array}{ll} a+b+c = 1 & (1) \\ a-b+d = 0 & (2) \\ a+b-c = 1 & (3) \\ a-b-d = 1 & (4) \end{array}$$

$(1)-(4):\ 2b+c+d=0$

$(2)-(4):\ 2d=-1 \Rightarrow \underline{\underline{d=-\frac{1}{2}}}$

$(3)-(4):\ 2b-c+d=0$

$$2b+c=\frac{1}{2}$$
$$-\ 2b-c=\frac{1}{2}$$

$2c=0 \Rightarrow \underline{\underline{c=0}}$

$2b=\frac{1}{2}$

$\underline{\underline{b=\frac{1}{4}}}$

$a=1-\frac{1}{4}-0$

$\underline{\underline{a=\frac{3}{4}}}$

$$\frac{x^3+x+1}{x^4-1} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}}{x^2+1}$$

$$I = \frac{3}{4}\int\frac{dx}{x-1} + \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{x^2+1} = \frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| - \frac{1}{2}\operatorname{arctg} x + C$$

5. $$\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx,$$

$x^2+1=0$ $\quad$ $x^2-x+1=0$

$D=-4<0$ $\quad$ $D=1-4<0$

x^2+1 i x^2-x+1 se ne mogu rastaviti

Rj. $$\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2-x+1} \quad /(x^2+1)(x^2-x+1)$$

$$x^3+3x-5 = (ax+b)(x^2-x+1) + (cx+d)(x^2+1)$$

$$x^3+3x-5 = a(x^3-x^2+x) + b(x^2-x+1) + c(x^3+x) + d(x^2+1)$$

$$x^3+3x-5 = (a+c)x^3 + (-a+b+d)x^2 + (a-b+c)x + (b+d)$$

$$\begin{array}{ll} a+c = 1 & (1) \\ -a+b+d = 0 & (2) \\ a-b+c = 3 & (3) \\ b+d = -5 & (4) \end{array}$$

$(1)\ a+c=1$

$(2)-(4)\ -a=5 \Rightarrow a=-5$

$(3)\ a-b+c=3$

$c=6$

$-5-b+6=3 \Rightarrow b=-2$

$\Rightarrow d=-3$

$$\frac{x^3-3x-5}{(x^2+1)(x^2-x+1)} = \frac{-5x-2}{x^2+1} + \frac{6x-3}{x^2-x+1}$$

$$I = \int\frac{-5x-2}{x^2+1}\,dx + 3\int\frac{2x-1}{x^2-x+1}\,dx = I_1 + 3I_2$$

$(x^2+1)' = 2x$

$-5x-2 = \alpha\cdot 2x + \beta$

$-5x-2 = 2\cdot\frac{-5}{2}x - 2$

$$I_1=\int\frac{2\cdot\frac{-5}{2}x-2}{x^2+1}dx=-\frac{5}{2}\int\frac{2x}{x^2+1}dx-2\int\frac{dx}{x^2+1}=-\frac{5}{2}\ln|x^2+1|-2\,\mathrm{arctg}(x^2+1)+C_1$$

$$I_2=\int\frac{2x-1}{x^2-x+1}dx=\left|\begin{matrix}x^2-x+1=t\\(2x-1)dx=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|x^2-x+1|+C_2$$

$$I=-\frac{5}{2}\ln|x^2+1|-2\,\mathrm{arctg}(x^2+1)+3\ln|x^2-x+1|+C$$

6. $$I=\int\frac{3}{x(x+1)^3}dx$$

Rj. $$\frac{3}{x(x+1)^3}=\frac{a}{x}+\frac{b}{x+1}+\frac{c}{(x+1)^2}+\frac{d}{(x+1)^3}\quad /\cdot x(x+1)^3$$

$$3=a(x+1)^3+bx(x+1)^2+cx(x+1)+dx$$

Ako uvrstimo $x=0$ u gornju jednakost dobićemo $3=a$

tj. $a=3$: $$3=3(x+1)^3+bx(x+1)^2+cx(x+1)+dx$$

za $x=-1$ imamo $3=d\cdot(-1)\Rightarrow d=-3$

$$3=3(x+1)^3+bx(x+1)^2+cx(x+1)-3x$$

za $x=1$: $\underline{3}=\underline{3\cdot 2^3}+b\cdot1\cdot2^2+c\cdot1\cdot2\underline{-3\cdot1}\Rightarrow 4b+2c=-18$

za $x=-2$: $3=3\cdot(-1)^3+b(-2)(-1)^2+c(-2)(-1)-3\cdot(-2)\Rightarrow -2b+2c=0$ $\Big\}+$

$\Rightarrow b=c=-3$

$$\int\frac{dx}{(x+1)^3}=\int(x+1)^{-3}dx=\frac{(x+1)^{-2}}{-2}+C_1=-\frac{1}{2(x+1)^2}+C_1$$

$$\frac{3}{x(x+1)^3}=\frac{3}{x}+\frac{-3}{x+1}+\frac{-3}{(x+1)^2}+\frac{-3}{(x+1)^3}$$

$$I=3\int\frac{dx}{x}-3\int\frac{dx}{x+1}-3\int\frac{dx}{(x+1)^2}-3\int\frac{dx}{(x+1)^3}=3\ln|x|-3\ln|x+1|+\frac{3}{x+1}+\frac{3}{2(x+1)^2}+C$$

7. ✓ $\displaystyle\int\frac{2x-3}{(x^2-3x+2)^3}dx$ Rj. $\displaystyle-\frac{1}{2(x^2-3x+2)^2}+C$

8. ✓ $\displaystyle\int\frac{x^3+x+1}{x(x^2+1)}dx$ Rj. $\displaystyle x+\ln\left|\frac{x}{\sqrt{x^2+1}}\right|+C$

9. ✓ $\displaystyle\int\frac{x^4}{x^4-1}dx$ Rj. $\displaystyle x+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\mathrm{arctg}\,x+C$

10. ✓ $\displaystyle\int\frac{dx}{(x^2-4x+3)(x^2+4x+5)}$ Rj. $\displaystyle\frac{1}{52}\ln|x-3|-\frac{1}{20}\ln|x-1|+\frac{1}{65}\ln(x^2+4x+5)+\frac{7}{130}\mathrm{arctg}(x+2)+C$

11. $I=\int \frac{-x^5-2x^2+2}{(x-1)(x^2+2x+1)}\,dx$

Rj. $(x-1)(x^2+2x+1)=x^3+2x^2+x-x^2-2x-1=x^3+x^2-x-1$

$$(-x^5-2x^2+2):(x^3+x^2-x-1)=-x^2+x-2-\frac{x}{x^3+x^2-x-1}$$

$$\begin{array}{l} -\;-x^5-x^4+x^3+x^2 \\ \hline /\!/\quad x^4-x^3-3x^2+2 \\ -\quad x^4+x^3-x^2-x \\ \hline /\!/\quad -2x^3-2x^2+x+2 \\ -\quad -2x^3-2x^2+2x+2 \\ \hline \qquad\qquad -x \end{array}$$

$$I=\int\left(-x^2+x-2-\frac{x}{(x-1)(x^2+2x+1)}\right)dx$$

$$=-\int x^2dx+\int x\,dx-2\int dx-\int\frac{x}{(x-1)(x+1)^2}dx$$

$=-\frac{x^3}{3}+\frac{x^2}{2}-2x-I_1$, integral I_1 smo odredili u zadatku 1

$$I=-\frac{x^3}{3}+\frac{x^2}{2}-2x-\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2(x+1)}+C$$

12. $I=\int \frac{x^5-2x^3+x-1}{x^3-2x^2+x}\,dx$

Rj. $(x^5-2x^3+x-1):(x^3-2x^2+x)=x^2+2x+1-\frac{1}{x^3-2x^2+x}$

$$\begin{array}{l} -\quad x^5-2x^4+x^3 \\ \hline /\!/\quad 2x^4-3x^3+x-1 \\ -\quad 2x^4-4x^3+2x^2 \\ \hline /\!/\quad x^3-2x^2+x-1 \\ -\quad x^3-2x^2+x \\ \hline \qquad\qquad -1 \end{array}$$

$$I=\int\left(x^2+2x+1-\frac{1}{x^3-2x^2+x}\right)dx$$

$$=\int x^2dx+2\int x\,dx+\int dx-\int\frac{dx}{x(x-1)^2}$$

$=\frac{x^3}{3}+2\cdot\frac{x^2}{2}+x-I_2$, integral I_2 smo odredili u zadatku 3.

$$I=\frac{x^3}{3}+x^2+x-\ln|x|+\ln|x-1|+\frac{1}{x-1}+C$$

13. $I=\int\frac{x^5+2x^3-4}{(x-2)(x^2+x+1)}\,dx$ Rj. $I=\frac{x^3}{3}+\frac{x^2}{2}+4x+\frac{44}{7}\ln|x-2|+\frac{5}{14}\ln|x^2+x+1|+\frac{11}{2\sqrt3}\,\mathrm{arctg}\,\frac{2x+1}{\sqrt3}+C$

14. $\displaystyle\int \frac{x^5-60x^3+73x^2+171}{x^2-9x+14}\,dx$

Rj. $(x^5-60x^3+73x^2+171):(x^2-9x+14) = x^3+9x^2+7x+10+\dfrac{-8x+31}{x^2-9x+14}$

$$\begin{array}{l}
-\ \underline{x^5-9x^4+14x^3} \\
=\quad 9x^4-74x^3+73x^2+171 \\
-\ \ \underline{9x^4-81x^3+126x^2} \\
=\qquad 7x^3-53x^2+171 \\
-\qquad \underline{7x^3-63x^2+98x} \\
=\qquad\quad 10x^2-98x+171 \\
-\qquad\quad \underline{10x^2-90x+140} \\
\qquad\qquad\quad -8x+31
\end{array}$$

$$I=\int\left(x^3+9x^2+7x+10+\frac{-8x+31}{x^2-9x+14}\right)dx$$

$$=\int x^3dx+9\int x^2dx+7\int x\,dx+10\int dx-\int\frac{8x-31}{x^2-9x+14}\,dx$$

$I=\dfrac{x^4}{4}+9\cdot\dfrac{x^3}{3}+7\dfrac{x^2}{2}+10x-I_3$, integral I_3 smo odredili u zadatku broj 2.

$$I=\frac{1}{4}x^4+3x^3+\frac{7}{2}x^2+10x-3\ln|x-2|+5\ln|x-7|+C$$

15. $\displaystyle I=\int \frac{x^7-2x^6+x^5+x^4+2x^2}{x^4-1}\,dx$

Rj. $(x^7-2x^6+x^5+x^4+2x^2):(x^4-1)=x^3-2x^2+x+1+\dfrac{x^3+x+1}{x^4-1}$

$$\begin{array}{l}
-\ \underline{x^7-x^3} \\
=\quad -2x^6+x^5+x^4+x^3+2x^2 \\
-\quad \underline{-2x^6+2x^2} \\
=\qquad x^5+x^4+x^3 \\
\qquad\quad \underline{x^5-x} \\
\qquad\qquad x^4+x^3+x \\
-\qquad\quad \underline{x^4-1} \\
\qquad\qquad\quad x^3+x+1
\end{array}$$

$$I=\int\left(x^3-2x^2+x+1+\frac{x^3+x+1}{x^4-1}\right)dx=$$

$$=\int x^3dx-2\int x^2dx+\int x\,dx+\int dx+\int\frac{x^3+x+1}{x^4-1}\,dx$$

$I=\dfrac{x^4}{4}-2\cdot\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+I_4$, integral I_4 smo odredili u zadatku 4

$$I=\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2+x+\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|-\frac{1}{2}\operatorname{arctg}x+C$$

16.^{v} $\displaystyle\int\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$ Rj. $I=\dfrac{x^2}{2}-2x+\dfrac{2}{x}+2\ln|x^2+2x+2|-$

$-2\operatorname{arctg}(x+1)+C$

17. $$\int \frac{-2x^7-x^6-x^3+6x^2-x}{(x^2+1)(x^2-x+1)}\,dx$$

Rj: $(x^2+1)\cdot(x^2-x+1)=x^4-x^3+x^2+x^2-x+1=x^4-x^3+2x^2-x+1$

$$(-2x^7-x^6-x^3+6x^2-x):(x^4-x^3+2x^2-x+1)=-2x^3-3x^2+x+5+\frac{x^3+3x-5}{x^4-x^3+2x^2-x+1}$$

$$\begin{array}{l}
-\ \ -2x^7+2x^6-4x^5+2x^4-2x^3 \\ \hline
=\ \ -3x^6+4x^5-2x^4+x^3+6x^2-x \\
-\ \ -3x^6+3x^5-6x^4+3x^3-3x^2 \\ \hline
=\ \ x^5+4x^4-2x^3+9x^2-x \\
-\ \ x^5-x^4+2x^3-x^2+x \\ \hline
=\ \ 5x^4-4x^3+10x^2-2x \\
-\ \ 5x^4-5x^3+10x^2-5x+5 \\ \hline
\ \ x^3+3x-5
\end{array}$$

$$I=\int\left(-2x^3-3x^2+x+5+\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\right)dx$$

$$=-2\int x^3\,dx-3\int x^2\,dx+\int x\,dx+5\int dx+\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx$$

$I=-2\cdot\frac{x^4}{4}-3\cdot\frac{x^3}{3}+\frac{x^2}{2}+5x+I_5$, integral I_5 smo odredili u zadatku broj 5

$$I=-\frac{1}{2}x^4-x^3+\frac{1}{2}x^2+5x-\frac{5}{2}\ln|x^2+1|-2\,\mathrm{arctg}(x^2+1)+3\ln|x^2-x+1|+C$$

18. $$\int\frac{x^5+x^4-8}{x^3-4x}$$

19. $$\int\frac{x^5+2}{x^3-1}\,dx$$